On the fractional metric dimension of graphs

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Abstract

In [S. Arumugam, V. Mathew and J. Shen, On fractional metric dimension of graphs, preprint], Arumugam et al. studied the fractional metric dimension of the cartesian product of two graphs, and proposed four open problems. In this paper, we determine the fractional metric dimension of vertex-transitive graphs, in particular, the fractional metric dimension of a vertex-transitive distance-regular graph is expressed in terms of its intersection numbers. As an application, we calculate the fractional metric dimension of Hamming graphs and Johnson graphs, respectively. Moreover, we give an inequality for metric dimension and fractional metric dimension of an arbitrary graph, and determine all graphs when the equality holds. Finally, we establish bounds on the fractional metric dimension of the cartesian product of graphs. As a result, we completely solve the four open problems.

Key words: resolving set; metric dimension; fractional metric dimension; vertex-transitive graph; distance-regular graph; cartesian product.

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1 Introduction

Let G be a finite, simple and connected graph. We often denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. For any two vertices x and y of G, $d_G(x, y)$ denotes the distance between x and y, $R_G\{x, y\}$ denotes the set of vertices z such that $d_G(x, z) \neq d_G(y, z)$. If the graph G is clear from the context, $d_G(x, y)$ and $R_G\{x, y\}$ will be written $d(x, y)$ and $R\{x, y\}$, respectively. A *resolving set* of G is a subset W of $V(G)$ such that $W \cap R_G\{x, y\} \neq \emptyset$ for any two distinct vertices x and y of G. The *metric dimension* of G, denoted by $\dim(G)$, is the minimum cardinality of all the resolving sets of G. Metric dimension was first introduced in the 1970s, independently by Harary and Melter [6] and by Slater [7]. It is a parameter that has appeared in various applications (see [3, 5] for more information).

Let $f\colon V(G) \to [0, 1]$ be a real value function. For $W \subseteq V(G)$, denote $f(W) = \sum_{v\in W} f(v)$. We call f a *resolving function* of G if $f(R_G\{x, y\}) \geq 1$ for any two

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distinct vertices x and y of G. The *fractional metric dimension*, denoted by $\dim_f(G)$, is given by

$$\dim_f(G) = \min\{|g| : g \text{ is a resolving function of } G\},$$

where $|g| = g(V(G))$. Arumugam and Mathew [1] formally introduced the fractional metric dimension of graphs and made some basic results.

The *cartesian product* of graphs G and H, denoted by $G\Box H$, is the graph with the vertex set $V(G)\times V(H) = \{(u,v)|u\in V(G), v\in V(H)\}$, where (u_1,v_1) is adjacent to (u_2,v_2) whenever $u_1=u_2$ and $\{v_1,v_2\}\in E(H)$, or $v_1=v_2$ and $\{u_1,u_2\}\in E(G)$. When there is no confusion the vertex (u,v) of $G\Box H$ will be written uv. Observe that $d_{G\Box H}(u_1v_1,u_2v_2) = d_G(u_1,u_2)+d_H(v_1,v_2)$.

Very recently, Arumugam et al. [2] characterized all graphs G satisfying $\dim_f(G) = \frac{|V(G)|}{2}$, presented several results on the fractional metric dimension of the cartesian product of graphs, and raised the following four open problems:

Problem 1. Determine $\dim_f(K_2\Box C_n)$ when n is odd, where K_2 is the complete graph of order 2 and C_n is a cycle of order n.

Problem 2. Determine $\dim_f(H_{n,k})$, where the Hamming graph $H_{n,k}$ is the cartesian product of n cliques K_k.

Problem 3. Cáceres et al. [5] proved $\dim(G\Box H)\geq \max\{\dim(G),\dim(H)\}$. Is a similar result true for $\dim_f(G\Box H)$?

Problem 4. Let G and H be two graphs with $\dim_f(G) = \frac{|V(G)|}{2}$ and $|V(H)|\leq |V(G)|$. Is $\dim_f(H\Box G) = \frac{|V(G)|}{2}$?

The motivation of this paper is to solve all these problems. In Section 2, we determine the fractional metric dimension of vertex-transitive graphs, in particular, the fractional metric dimension of a vertex-transitive distance-regular graph is expressed in terms of its intersection numbers. As an application, we calculate the fractional metric dimension of Hamming graphs and Johnson graphs, respectively. In Section 3, we give an inequality for metric dimension and fractional metric dimension of an arbitrary graph, and determine all graphs when the equality holds. In Section 4, we establish bounds on the fractional metric dimension of the cartesian product of graphs.

2 Vertex-transitive graphs

For a graph G, in this paper we always assume that

$$r(G) = \min\{|R\{x,y\}| \mid x,y\in V(G), x\neq y\}. \tag{1}$$

In this section we shall express the fractional metric dimension of a vertex-transitive graph G in terms of the parameter $r(G)$, and solve Problems 1 and 2.

Lemma 2.1 *Let G be a graph with $r(G)$ as in (1). Then $\dim_f(G)\leq \frac{|V(G)|}{r(G)}$.*

Proof. Define $f : V(G)\longrightarrow [0,1]$, $x\longrightarrow \frac{1}{r(G)}$. For any two distinct vertices x and y, we have

$$f(R\{x,y\}) = \frac{|R\{x,y\}|}{r(G)}\geq 1,$$

which implies that f is a resolving function. Hence, $\dim_f(G) \le |f| = \frac{|V(G)|}{r(G)}$. $\square$

A graph G is *vertex-transitive* if its full automorphism group $\mathrm{Aut}(G)$ acts transitively on the vertex set.

Theorem 2.2 *Let G be a vertex-transitive graph with $r(G)$ as in* (1). *Then* $\dim_f(G) = \frac{|V(G)|}{r(G)}$.

Proof. Denote $r = r(G)$. Then there exist two distinct vertices u and v such that $|R\{u,v\}| = r$. Suppose $R\{u,v\} = \{w_1, \ldots, w_r\}$. For any automorphism σ of G,

$$R\{\sigma(u), \sigma(v)\} = \{\sigma(w_1), \ldots, \sigma(w_r)\}.$$

Let f be a resolving function with $\dim_f(G) = |f|$. Then

$$f(\sigma(w_1)) + \cdots + f(\sigma(w_r)) = f(R\{\sigma(u), \sigma(v)\}) \ge 1,$$

which implies that

$$\sum_{\sigma \in \mathrm{Aut}(G)} (f(\sigma(w_1)) + \cdots + f(\sigma(w_r))) \ge |\mathrm{Aut}(G)|.$$

Since G is vertex transitive, we have

$$|\mathrm{Aut}(G)_{w_1}| \cdot |f| + \cdots + |\mathrm{Aut}(G)_{w_r}| \cdot |f| \ge |\mathrm{Aut}(G)|.$$

It follows that $\dim_f(G) = |f| \ge \frac{|V(G)|}{r}$. By Lemma 2.1 we accomplish our proof. $\square$

Arumugam et al. [2] proved that $\dim_f(K_2 \square C_n) = 2$ when n is even. Here we consider the remaining case.

Theorem 2.3 *If n is an odd integer with $n \ge 3$, then* $\dim_f(K_2 \square C_n) = \frac{2n}{n+1}$.

Proof. For any two distinct vertices u_1v_1 and u_2v_2 of $K_2 \square C_n$, we have

$$|R\{u_1v_1, u_2v_2\}| = \begin{cases} 2n-2, & \text{if } u_1 = u_2, v_1 \ne v_2, \\ 2n, & \text{if } u_1 \ne u_2, v_1 = v_2, \\ n+1, & \text{if } u_1 \ne u_2, d_{C_n}(v_1, v_2) = 1, \\ 2n-2, & \text{if } u_1 \ne u_2, d_{C_n}(v_1, v_2) \ge 2. \end{cases}$$

Since $K_2 \square C_n$ is vertex-transitive, $\dim_f(K_2 \square C_n) = \frac{2n}{n+1}$ by Theorem 2.2. $\square$

Next we shall consider the fractional metric dimension of distance-regular graphs, in particular we compute this parameter of Hamming graphs and Johnson graphs, respectively.

A graph G with diameter d is said to be *distance-regular* if, for all integers $0 \le h, i, j \le d$ and any two vertices x, y at distance h, the number

$$p_{i,j}^h = |\{z \in V(G) \mid d(x,z) = i, d(y,z) = j\}|$$

is a constant. The numbers $p_{i,j}^h$ are called the *intersection numbers* of G. For more information about distance-regular graphs, we would like to refer readers to [4].

Theorem 2.4 *Let G be a vertex-transitive distance-regular graph with diameter d. Then*

$$\dim_f(G) = \frac{|V(G)|}{|V(G)| - \max\{\sum_{i=1}^{d} p_{i,i}^h | h = 1, \dots, d\}}.$$

Proof. For any two distinct vertices x and y at distance h, $|R\{x, y\}| = |V(G)| - \sum_{i=1}^{d} p_{i,i}^h$. By Theorem 2.2, the desired result follows. □

The *Hamming graph*, denoted by $H_{n,k}$, has the vertex set $\{(x_1, \dots, x_n) | 1 \le x_i \le k, 1 \le i \le n\}$, with two vertices being adjacent if they differ in exactly one coordinate. It is well-known that $H_{n,k}$ is a vertex-transitive distance-regular graph of order k^n, and two vertices are at distance j if and only if they differ in exactly j coordinates. The *hypercube* Q_n is the Hamming graph $H_{n,2}$. Arumugam and Mathew [1] proved $\dim_f(Q_n) = 2$ for $n \ge 2$. Now we compute $\dim_f(H_{n,k})$.

Theorem 2.5 *Let $H_{n,k}$ be a Hamming graph where $k \ge 3$. Then $\dim_f(H_{n,k}) = \frac{k}{2}$.*

Proof. Let $\delta_{i,j}$ denote the Kronecker delta. Pick two vertices

$$u = (1, \dots, 1), \quad v = (\underbrace{2, \dots, 2}_{h}, 1, \dots, 1).$$

Then $d(u, v) = h$. Since, for any vertex $w = (w_1, \dots, w_n)$, $d(u, w) = d(v, w)$ if and only if $\sum_{i=1}^{h} \delta_{1,w_i} = \sum_{i=1}^{h} \delta_{2,w_i}$, then the intersection numbers of $H_{n,k}$ satisfy

$$\sum_{i=1}^{n} p_{i,i}^h = \sum_{s=0}^{\lfloor \frac{h}{2} \rfloor} \binom{h}{2s} \binom{2s}{s} (k-2)^{h-2s} k^{n-h}. \tag{2}$$

Since $\sum_{i=1}^{n} p_{i,i}^1 = (k-2)k^{n-1}$, by Theorem 2.4 it suffices to show that

$$\sum_{i=1}^{n} p_{i,i}^h \le (k-2)k^{n-1}, \quad 2 \le h \le n.$$

For $1 \le s \le \frac{h}{2}$, we have

$$\begin{aligned} \binom{h}{2s}\binom{2s}{s} &= \binom{h-1}{2s-1}\binom{2s}{s} + \binom{h-1}{2s}\binom{2s}{s} \\ &\le \binom{h-1}{2s-1} \cdot 2^{2s-1} + \binom{h-1}{2s} \cdot 2^{2s}, \end{aligned}$$

which implies that

$$\begin{aligned} &\sum_{s=0}^{\lfloor \frac{h}{2} \rfloor} \binom{h}{2s}\binom{2s}{s}(k-2)^{-2s} \\ \le\ & 1 + \sum_{s=1}^{\lfloor \frac{h}{2} \rfloor} \left(\binom{h-1}{2s-1} \left(\frac{2}{k-2}\right)^{2s-1} + \binom{h-1}{2s} \left(\frac{2}{k-2}\right)^{2s} \right) \\ =\ & \left(1 + \frac{2}{k-2}\right)^{h-1}. \end{aligned}$$

By (2), we get

$$\sum_{i=1}^{n} p_{i,i}^{h} \le \big(1+\frac{2}{k-2}\big)^{h-1}(k-2)^h k^{n-h} = (k-2)k^{n-1},$$

as desired. $\square$

Let X be a set of size n, and let $\binom{X}{k}$ denote the set of all k-subsets of X. The *Johnson graph*, denoted by $J(n,k)$, has $\binom{X}{k}$ as the vertex set, where two k-subsets are adjacent if their intersection has size $k-1$. As we know, $J(n,k)$ is a vertex-transitive distance-regular graph of order $\binom{n}{k}$, and two vertices are at distance j if and only if their intersection has size $k-j$. Since $J(n,k) \simeq J(n,n-k)$ and $J(n,1) \simeq K_n$, we only consider the case $4 \le 2k \le n$. In order to calculate $\dim_f(J(n,k))$, we need the following result, the proof of which is immediate from the unimodality of binomial coefficients.

Lemma 2.6 *Let m be a positive integer and n be an arbitrary integer. Then*

$$\binom{m}{n+1} + \binom{m}{n-1} \ge \binom{m}{n}.$$

Theorem 2.7 *Let $J(n,k)$ be a Johnson graph with $4 \le 2k \le n$. Then*

$$\dim_f(J(n,k)) = \begin{cases} 3, & \text{if } (n,k)=(4,2),\\ \frac{35}{17}, & \text{if } (n,k)=(8,4),\\ \frac{n^2-n}{2kn-2k^2}, & \text{otherwise.} \end{cases}$$

Proof. Pick any two distinct vertices A and B at distance h, write $A_1 = A\setminus(A\cap B)$ and $B_1 = B\setminus(A\cap B)$. Then $|A_1| = |A_2| = h$ and $A_1 \cap B_1 = \emptyset$. Since, for any vertex C, $d(A,C) = d(B,C)$ if and only if $|A_1 \cap C| = |B_1 \cap C|$, then the intersection numbers of $J(n,k)$ satisfy

$$\sum_{i=1}^{k} p_{i,i}^{h} = \sum_{s=0}^{h} \binom{h}{s}^2 \binom{n-2h}{k-2s}. \tag{3}$$

If $(n,k) = (4,2)$, by Theorem 2.4 we have $\dim_f(J(4,2)) = 3$. If $(n,k) = (8,4)$, by (3) and Theorem 2.4 we obtain $\dim_f(J(8,4)) = \frac{35}{17}$.

Now suppose $(n,k) \notin \{(4,2),(8,4)\}$. Since $\sum_{i=1}^{k} p_{i,i}^{1} = \binom{n-2}{k} + \binom{n-2}{k-2}$, by Theorem 2.4 it suffices to show that for $2 \le h \le k \le \frac{n}{2}$,

$$\sum_{i=1}^{k} p_{i,i}^{h} \le \binom{n-2}{k} + \binom{n-2}{k-2}. \tag{4}$$

We divide our proof into two cases.

Case 1. $h = \frac{n}{2}$. Then $h = k$. By (3), we have

$$\sum_{i=1}^{k} p_{i,i}^{h} = \begin{cases} 0, & k \text{ is odd},\\ \binom{k}{\frac{k}{2}}^2, & k \text{ is even.} \end{cases}$$

Since

$$\binom{k}{\frac{k}{2}}^2 \leq 2\binom{2k-2}{k-2} = \binom{n-2}{k} + \binom{n-2}{k-2} \quad \text{for } k \geq 6,$$

then (4) holds.

Case 2. $2 \leq h < \frac{n}{2}$. For $1 \leq s \leq h-1$ and $0 \leq j \leq 2$, we have

$$\binom{2h-2}{2s-j} = \sum_{i=0}^{h-2}\binom{h-2}{i}\binom{h}{2s-j-i} \geq \binom{h-2}{s-j}\binom{h}{s} + \binom{h-2}{s-1}\binom{h}{s-j+1}. \quad (5)$$

By Lemma 2.6 and (5), we have

$$\begin{aligned}
& \sum_{i=1}^{2h-2}\binom{2h-2}{i}\binom{n-2h}{k-i} + \sum_{i=0}^{2h-3}\binom{2h-2}{i}\binom{n-2h}{k-i-2} \\
= & \sum_{s=1}^{h-1}\left\{\left[\binom{2h-2}{2s} + \binom{2h-2}{2s-2}\right]\binom{n-2h}{k-2s}\right. \\
& \quad \left. + \binom{2h-2}{2s-1}\left[\binom{n-2h}{k-2s+1} + \binom{n-2h}{k-2s-1}\right]\right\} \\
\geq & \sum_{s=1}^{h-1}\left[\binom{2h-2}{2s} + \binom{2h-2}{2s-2} + \binom{2h-2}{2s-1}\right]\binom{n-2h}{k-2s} \\
\geq & \sum_{s=1}^{h-1}\left[\binom{h-2}{s} + 2\binom{h-2}{s-1} + \binom{h-2}{s-2}\right]\binom{h}{s}\binom{n-2h}{k-2s} \\
= & \sum_{s=1}^{h-1}\binom{h}{s}^2\binom{n-2h}{k-2s}.
\end{aligned}$$

Then

$$\begin{aligned}
\binom{n-2}{k} + \binom{n-2}{k-2} &= \sum_{i=0}^{2h-2}\binom{2h-2}{i}\binom{n-2h}{k-i} + \sum_{i=0}^{2h-2}\binom{2h-2}{i}\binom{n-2h}{k-i-2} \\
&\geq \binom{n-2h}{k} + \sum_{s=1}^{h-1}\binom{h}{s}^2\binom{n-2h}{k-2s} + \binom{n-2h}{k-2h} \\
&= \sum_{s=0}^{h}\binom{h}{s}^2\binom{n-2h}{k-2s}.
\end{aligned}$$

Hence (4) holds by (3). $\square$

3 An inequality

In this section, we give an inequality for metric dimension and fractional metric dimension of any graph, and determine all graphs when the equality holds.

Lemma 3.1 *Let G be a graph. For any subset A of $V(G)$ with size $|V(G)| - \dim(G) + 1$, there exist two distinct vertices x and y of G such that $R\{x, y\} \subseteq A$.*

Proof. Suppose there exists a subset A with size $|V(G)| - \dim(G) + 1$ such that $R\{x, y\} \nsubseteq A$ for any two distinct vertices x and y. Then $R\{x, y\} \cap (V(G) \setminus A) \neq \emptyset$; and so $V(G) \setminus A$ is a resolving set of G. Therefore, $\dim(G) - 1 = |V(G) \setminus A| \geq \dim(G)$, a contradiction. $\square$

Lemma 3.2 *Let G be a graph with $r(G)$ as in* (1)*. Then $r(G) = |V(G)| - 1$ if and only if G is isomorphic to a path or an odd cycle.*

Proof. The sufficiency is immediate. Conversely, suppose $r(G) = |V(G)| - 1$. Denote the maximum degree of G by Δ. Pick a vertex x with degree Δ. Suppose $\Delta \geq 3$. We may choose three pairwise distinct vertices x_1, x_2 and x_3 adjacent to x. Observe $x \notin R\{x_1, x_2\} \cup R\{x_1, x_3\} \cup R\{x_2, x_3\}$. Then $x_3 \in R\{x_1, x_2\}$, which implies that $d(x_3, x_1) \neq d(x_3, x_2)$. We may assume $d(x_3, x_1) = 1$ and $d(x_3, x_2) = 2$. From $x_2 \in R\{x_1, x_3\}$ we get $d(x_1, x_2) = 1$, which implies that $x_1 \notin R\{x_2, x_3\}$, a contradiction. Hence $\Delta \leq 2$; and so G is isomorphic to a path or a cycle. If G is isomorphic to an even cycle, then for two vertices u and v at distance 2, we have $|R\{u, v\}| = n - 2$, a contradiction. Hence, the desired result follows. $\square$

Lemma 3.3 *Let G be a graph. Suppose $|\overline{R}\{u, v\}| = 2$ for any two distinct vertices u and v, where $\overline{R}\{u, v\} = V(G) \setminus R\{u, v\}$. If the map*

$$\varphi : \binom{V(G)}{2} \longrightarrow \binom{V(G)}{2}, \quad \{u, v\} \longmapsto \overline{R}\{u, v\}$$

is a bijection, then G is isomorphic to the complete graph of order four.

Proof. For any two adjacent vertices x and y, let $D^i_j(x, y) = \{u \in V(G) \mid d(x, u) = i, d(y, u) = j\}$. The intersection diagram with respect to x and y is the collection $\{D^i_j(x, y)\}_{i,j}$ with lines between $D^i_j(x, y)$'s and $D^s_t(x, y)$'s. We draw a line between $D^i_j(x, y)$ and $D^s_t(x, y)$ if there is possibility of existence of edges. The intersection diagram with respect to x and y is shown in Figure 1, where $D^i_j = D^i_j(x, y)$ and d is the diameter of G.

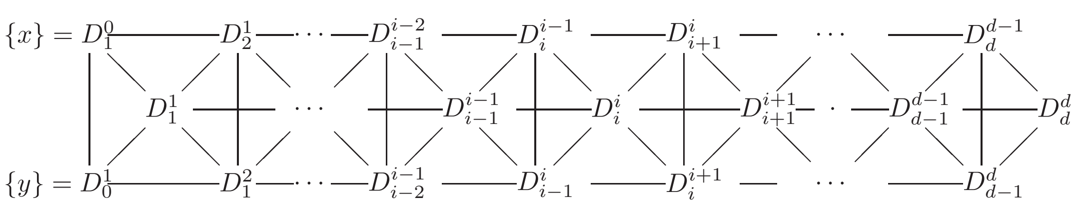

Figure 1: Intersection diagram with respect to x and y.

Since $\overline{R}\{x, y\} = \cup_{i=1}^d D^i_i$, then

$$\sum_{i=1}^{d} |D^i_i| = 2. \tag{6}$$

Note that two distinct vertices u and v belong to some D_j^i if and only if $\overline{R}\{u,v\} = \{x,y\}$. Since φ is a bijection, there exist a and b such that $|D_b^a| = 2$ and

$$|D_j^i| \leq 1 \text{ for } (i,j) \neq (a,b). \tag{7}$$

Write $D_b^a = \{z_1, z_2\}$.

Claim. There exist two adjacent vertices x_0 and y_0 such that $|D_1^1(x_0,y_0)| = 2$.

Suppose that, for any two adjacent vertices x and y,

$$|D_1^1(x,y)| \leq 1. \tag{8}$$

Case 1. $a = b$. By (6), $D_i^i = \emptyset$ for $i \neq a$. Then there exist $x' \in D_a^{a-1}$ and $y' \in D_{a-1}^a$ such that $\overline{R}\{x',y'\} = \{z_1,z_2\} = \overline{R}\{x,y\}$. Hence $\{x',y'\} = \{x,y\}$ and $|D_1^1| = 2$, contrary to (8).

Case 2. $a \neq b$. By (7), there exists a unique $z \in D_{b-1}^{a-1}$ such that $z \in \overline{R}\{z_1,z_2\} = \{x,y\}$. Therefore, $(a,b) = (1,2)$ or $(2,1)$. We may assume $D_2^1 = \{z_1,z_2\}$. By (6) and (7), there exist $i_0 < j_0$ such that $|D_{i_0}^{i_0}| = |D_{j_0}^{j_0}| = 1$ and $D_i^i = \emptyset$ for $i \neq i_0, j_0$. Write $D_{i_0}^{i_0} = \{w_1\}$ and $D_{j_0}^{j_0} = \{w_2\}$. Since $w_1 \notin \overline{R}\{z_1,z_2\}$, then $d(w_1,z_1) \neq d(w_1,z_2)$. Without loss of generality, assume that $d(w_1,z_1) < d(w_1,z_2)$.

Case 2.1. $i_0 = 1$. Then $d(w_1,z_1) = 1$, which implies $D_1^1(x,w_1) = \{y,z_1\}$. Consequently, x, w_1 are adjacent and $|D_1^1(x,w_1)| = 2$, contrary to (8).

Case 2.2. $i_0 = 2$. Note that $D_1^1 = \emptyset$. Then $d(w_1,z_1) = d(w_1,y_1) = 1$, where y_1 is the unique vertex in D_1^2. If $D_3^2 = \emptyset$ or $D_2^3 = \emptyset$, then $j_0 = 3$ and $d(w_1,w_2) = 1$. Consequently, $\overline{R}\{z_1,y_1\} = \{w_1,w_2\} = \overline{R}\{x,y\}$, which contradicts the fact that φ is a bijection. Write $D_3^2 = \{x_2\}$ and $D_2^3 = \{y_2\}$. Since $\overline{R}\{z_1,y_1\} \neq \{w_1,w_2\}$, then $d(w_2,z_1) \neq d(w_2,y_1)$. The fact that $d(w_2,x_2) = d(w_2,y_2)$ implies that x_2 is not adjacent to z_1 and $d(x_2,z_2) = 1$; and so $V(G) \setminus \{z_1,w_1,w_2\} \subseteq R\{x_2,y_2\} \cap R\{z_2,y_1\}$. Then $\overline{R}\{x_2,y_2\} \cup \overline{R}\{z_2,y_1\} \subseteq \{z_1,w_1,w_2\}$. Since $w_2 \in \overline{R}\{x_2,y_2\}$ and $\overline{R}\{x,y\} = \{w_1,w_2\}$, we get $\overline{R}\{x_2,y_2\} = \{z_1,w_2\}$. Consequently, $\overline{R}\{z_2,y_1\} = \{z_1,w_1\}$, which implies $d(w_1,z_2) = d(w_1,y_1) = d(w_1,z_1)$. Hence $w_1 \in \overline{R}\{z_1,z_2\}$, which contradicts $\overline{R}\{z_1,z_2\} = \{x,y\}$.

Case 2.3. $i_0 \geq 3$. Note that $D_{i_0-1}^{i_0-1} = \emptyset$. By (7), $|D_{i_0}^{i_0-1}| = |D_{i_0-1}^{i_0}| = 1$. Write $D_{i_0}^{i_0-1} = \{x'\}$ and $D_{i_0-1}^{i_0} = \{y'\}$. Since $d(x',w_1) = d(y',w_1)$ and $d(x',w_2) = d(y',w_2)$, then $\overline{R}\{x',y'\} = \{w_1,w_2\} = \overline{R}\{x,y\}$, a contradiction.

Therefore, our claim is valid.

Now write $D_1^1(x_0,y_0) = \{z_1',z_2'\}$. By (6), $D_i^i(x_0,y_0) = \emptyset$ for $i \geq 2$. By (7), $|D_j^i(x_0,y_0)| \leq 1$ for $i \neq j$.

Suppose $D_2^1(x_0,y_0) \cup D_1^2(x_0,y_0) \neq \emptyset$. We may assume that $D_2^1(x_0,y_0) \neq \emptyset$. Write $D_2^1(x_0,y_0) = \{x_1\}$. If $D_1^2(x_0,y_0) = \emptyset$, then $V(G) \setminus \{z_1',z_2'\} \subseteq R\{x,x_1\}$, which implies that $\overline{R}\{x,x_1\} = \{z_1',z_2'\} = \overline{R}\{x,y\}$, a contradiction. If $D_1^2(x_0,y_0) \neq \emptyset$, write $D_1^2(x_0,y_0) = \{y_1\}$, then $V(G) \setminus \{z_1',z_2'\} \subseteq R\{x_1,y_1\}$. Hence, $\overline{R}\{x_1,y_1\} = \{z_1',z_2'\}$, a contradiction. Consequently, $D_2^1(x_0,y_0) = D_1^2(x_0,y_0) = \emptyset$, and $|V(G)| = 4$. Since $\overline{R}\{x_0,z_1'\} = \{y_0,z_2'\}$, we have $d(z_1',z_2') = 1$; and then $G \simeq K_4$. $\square$

Now we state our main result of this section.

Theorem 3.4 *Let G be a graph of order n. Then*

$$\dim_f(G) \geq \frac{n}{n - \dim(G) + 1}. \tag{9}$$

Moreover, the equality holds if and only if G is isomorphic to a path, a complete graph, or an odd cycle.

Proof. Write $l = n - \dim(G) + 1$. Suppose f is a resolving function of G with $|f| = \dim_f(G)$. By Lemma 3.1, $f(A) \geq 1$ for each $A \in \binom{V(G)}{l}$. Hence $\sum_{A \in \binom{V(G)}{l}} f(A) \geq \binom{n}{l}$. Since $\sum_{A \in \binom{V(G)}{l}} f(A) = \binom{n-1}{l-1}|f|$, then (9) holds.

Suppose that the equality in (9) holds. Then $f(A) = 1$ for each $A \in \binom{V(G)}{l}$. If $\dim(G) = 1$, then $G \simeq P_n$. If $\dim(G) = n - 1$, then $G \simeq K_n$. Now suppose $2 \leq \dim(G) \leq n - 2$. Then $3 \leq l \leq n - 1$.

Given two distinct vertices x, y, pick an $(l-1)$-subset A_1 of $V(G) \setminus \{x, y\}$. Since $f(\{x\} \cup A_1) = 1 = f(\{y\} \cup A_1)$, then $f(x) = f(y) = \frac{1}{l}$, which implies that $|R\{x, y\}| \geq l$. By Lemma 3.1, for any $A \in \binom{V(G)}{l}$, there exist two distinct vertices x_0 and y_0 such that $R\{x_0, y_0\} = A$. Hence $r(G) = l$, and

$$\binom{n}{l} \leq |\{R\{u, v\} | u \neq v\}| \leq \binom{n}{2}. \tag{10}$$

It follows that $l = n - 1$ or $l = n - 2$.

Case 1. $l = n - 1$. By Lemma 3.2, G is isomorphic to an odd cycle.

Case 2. $l = n - 2$. In this case $n \geq 5$. By (10), we have $|\{R\{u, v\} | u \neq v\}| = \binom{n}{l}$. By Lemma 3.1 we get $|R\{x, y\}| = l$ for any two distinct vertices x and y. Then we obtain a bijection φ as in Lemma 3.3. Hence, $G \simeq K_4$, a contradiction.

The converse is true by [1, Corollary 2.7 and Theorem 3.2]. $\square$

Combining Lemma 2.1 and Theorem 3.4, we obtain the following corollary.

Corollary 3.5 *Let G be a distance-regular graph with diameter d. Then*

$$\dim(G) \leq \max\{\sum_{i=1}^{d} p_{i,i}^h | h = 1, \ldots, d\} + 1.$$

The equality holds if and only if G is a complete graph or an odd cycle.

4 Cartesian product of graphs

In this section, we shall establish bounds on the fractional metric dimension of the cartesian product of two graphs.

Theorem 4.1 *Let G and H be two graphs. Then* $\dim_f(G \Box H) \geq \dim_f(G)$.

Proof. Pick a resolving function $f_{G \Box H}$ of $G \Box H$ with $|f_{G \Box H}| = \dim_f(G \Box H)$. Define

$$f_G : V(G) \longrightarrow [0, 1], \quad u \longmapsto \min\{1, \sum_{y \in V(H)} f_{G \Box H}(uv)\}.$$

Let u_1 and u_2 be two distinct vertices of G. We shall prove

$$f_G(R_G\{u_1,u_2\}) \geq 1. \tag{11}$$

If there exists $u_0 \in R_G\{u_1,u_2\}$ with $f_G(u_0) = 1$, then (11) holds. Now we suppose $f_G(u) = \sum_{v\in V(H)} f_{G\Box H}(uv)$ for any $u \in V(G)$. For $v_0 \in V(H)$, we have

$$R\{u_1v_0, u_2v_0\} = \bigcup_{u\in R_G\{u_1,u_2\}} \bigcup_{v\in V(H)} \{uv\}.$$

Then

$$f_G(R_G\{u_1,u_2\}) = \sum_{u\in R_G\{u_1,u_2\}} \sum_{v\in V(H)} f_{G\Box H}(uv) = f_{G\Box H}(R\{u_1v_0,u_2v_0\}) \geq 1,$$

(11) holds. Therefore, f_G is a resolving function of G. Since

$$|f_G| \leq \sum_{u\in V(G)} \sum_{v\in V(H)} f_{G\Box H}(uv) = |f_{G\Box H}|,$$

then $\dim_f(G) \leq \dim_f(G\Box H)$, as desired. $\Box$

Since $G\Box H$ is isomorphic to $H\Box G$, this theorem gives an answer to Problem 3. By Theorem 2.5 the bound in Theorem 4.1 is sharp.

Theorem 4.2 *Let G and H be two graphs. Then*

$$\dim_f(G\Box H) \leq \max\{\dim_f(G), |V(H)|\}.$$

Proof. Let f_G be a resolving function of G with $|f_G| = \dim_f(G)$. Denote $l = \min\{\dim_f(G), |V(H)|\}$. Define

$$f_{G\Box H} : V(G\Box H) \longrightarrow [0,1], \quad uv \longmapsto \frac{f_G(u)}{l}.$$

For any two distinct vertices u_1v_1 and u_2v_2 in $G\Box H$, we shall prove

$$f_{G\Box H}(R\{u_1v_1,u_2v_2\}) \geq 1.$$

Case 1. $v_1 = v_2$. Since

$$R\{u_1v_1, u_2v_1\} = \bigcup_{u\in R_G\{u_1,u_2\}} \bigcup_{v\in V(H)} \{uv\},$$

then

$$f_{G\Box H}(R\{u_1v_1,u_2v_1\}) = \sum_{u\in R_G\{u_1,u_2\}} \sum_{v\in V(H)} \frac{f_G(u)}{l} = \frac{|V(H)|}{l} \cdot f_G\{R_G\{u_1,u_2\}\} \geq 1.$$

Case 2. $v_1 \neq v_2$. Write

$$\begin{aligned} S_1 &= \{u \in V(G) \mid d_G(u_1,u) = d_G(u_2,u)\}, \\ S_2 &= \{u \in V(G) \mid d_G(u_1,u) < d_G(u_2,u)\}, \\ S_3 &= \{u \in V(G) \mid d_G(u_1,u) > d_G(u_2,u)\}. \end{aligned}$$

Then

$$R\{u_1v_1,u_2v_2\}\supseteq\Big(\bigcup_{u\in S_1\cup S_2}\{uv_1\}\Big)\cup\Big(\bigcup_{u\in S_1\cup S_3}\{uv_2\}\Big).$$

It follows that

$$\begin{aligned}
f_{G\Box H}(R\{u_1v_1,u_2v_2\}) &\geq \sum_{u\in S_1\cup S_2} f_{G\Box H}(uv_1)+\sum_{u\in S_1\cup S_3} f_{G\Box H}(uv_2)\\
&= \frac{f_G(S_1\cup S_2)}{l}+\frac{f_G(S_1\cup S_3)}{l}\\
&= \frac{\dim_f(G)}{l}+\frac{f_G(S_1)}{l}\\
&\geq 1.
\end{aligned}$$

Therefore, $f_{G\Box H}$ is a resolving function of $G\Box H$. Since

$$|f_{G\Box H}|=\sum_{v\in V(H)}\sum_{u\in V(G)}\frac{f_G(u)}{l}=\sum_{v\in V(H)}\frac{\dim_f(G)}{l}=\max\{\dim_f(G),|V(H)|\},$$

the desired result follows. □

By [1, Theorem 4.2] and [2, Theorem 3.3], $P_n\Box K_2$ and $C_{2n}\Box K_2$ meet the bound in Theorem 4.2.

Finally, we focus on Problem 4.

Theorem 4.3 *Let G be a graph with at least three vertices and $\dim_f(G)=\frac{|V(G)|}{2}$. Let H be a graph with $|V(H)|\leq|V(G)|$. Then $\dim_f(G\Box H)=\frac{|V(G)|}{2}$.*

Proof. By Theorem 4.1, $\dim_f(G\Box H)\geq\frac{|V(G)|}{2}$. In order to prove $\dim_f(G\Box H)\leq\frac{|V(G)|}{2}$, by Lemma 2.1 it suffices to show that

$$|R\{u_1v_1,u_2v_2\}|\geq 2|V(H)| \tag{12}$$

holds for any two distinct vertices u_1v_1 and u_2v_2 in $G\Box H$.

Case 1. $u_1=u_2$. Since $R\{u_1v_1,u_1v_2\}\supseteq\{uv_1|u\in V(G)\}\cup\{uv_2|u\in V(G)\}$, then $|R\{u_1v_1,u_1v_2\}|\geq 2|V(G)|\geq 2|V(H)|$, (12) holds.

Case 2. $u_1\neq u_2$. For $v\in V(H)$, let

$$S_v=\{u|u\in V(G),d_G(u_1,u)-d_G(u_2,u)\neq k_v\},$$

where $k_v=d_H(v_2,v)-d_H(v_1,v)$. Note that $R\{u_1v_1,u_2v_2\}=\cup_{v\in V(H)}\{uv|u\in S_v\}$. In order to prove (12), we only need to show that

$$|S_v|\geq 2. \tag{13}$$

Case 2.1. $k_v\neq d_G(u_1,u_2)$ and $k_v\neq -d_G(u_1,u_2)$. Then $u_1,u_2\in S_v$, and (13) holds.

Case 2.2. $k_v=d_G(u_1,u_2)$. Then $u_1\in S_v$. Since $\dim_f(G)=\frac{|V(G)|}{2}$, by [2, Theorem 2.2] there exists a vertex $u_2'\in V(G)\setminus\{u_2\}$ such that, for any $u\in V(G)\setminus\{u_2,u_2'\}$,

$$d_G(u_2',u)=d_G(u_2,u). \tag{14}$$

If $u_2' \neq u_1$, by (14) we have $d_G(u_1, u_2') - d_G(u_2, u_2') < d_G(u_1, u_2') = k_v$, which implies $u_2' \in S_v$ and (13) holds. Now suppose $u_2' = u_1$. Choose $u_3 \in V(G) \setminus \{u_1, u_2\}$. By (14), $d_G(u_1, u_3) - d_G(u_2, u_3) = 0 < k_v$. Then $u_3 \in S_v$, and so (13) holds.

Case 2.3. $k_v = -d_G(u_1, u_2)$. Similar to Case 2.2, (13) holds. $\square$

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